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Voltage distribution in growing conducting networks

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Abstract. We investigate by random-walk simulations and a mean-field theory how growth by biased addition of nodes affects flow of the current through the emergent conducting graph, representing a digital circuit. In the interior of a large network the voltage varies with the addition time $s < t$ of the node as $V(s) \sim \ln(s)/s^{\theta}$ when constant current enters the network at last added node t and leaves at the root of the graph which is grounded. The topological closeness of the conduction path and shortest path through a node suggests that the charged random walk determines these global graph properties by using only *local* search algorithms. The results agree with mean-field theory on tree structures, while the numerical method is applicable to graphs of any complexity.

PACS. 89.75.Hc Networks and genealogical trees – 05.40.Fb Random walks and Levy flights – 89.20.-a Interdisciplinary applications of physics

Networks, which are adequately represented by random graphs, invade all sciences [1–3]. In a classical approach, random graph theory deals with linking in a static graph with a given number of nodes [4]. Recently dynamically evolving networks came into focus [2,3] representing connections in complex dynamical systems, *e.g.*, metabolic or protein networks, and realistic social and technological networks, Internet and the Web, which are not static but evolve in time. Details of the growth rules, in which new and/or preexisting nodes are linked to the network, determine the graph topology that emerges after long evolution time. In the class of scale-free networks preference attachment rules lead to power-law degree distributions of incoming and outgoing links [2,3,5].

Conducting networks, such as electrical or electronic circuits are of particular importance for technology. A common technological network—digital circuit—consists of logic gates, as nodes, and wires in a broad sense, as links [6]. Technology advance with integration and miniaturization allows digital circuits to grow in size and complexity in order to optimize their function and stability. It was shown recently [6] that electronic circuits exhibit a scale-free link structure up to a cut-off size. So far, electrical properties of growing conducting networks have not received much attention in the literature.

Here we adapt the random-walk dynamics and mean-field theory to study for the first time how the growth of a conducting network interferes with the current flow through the underlying *evolving graph*. In particular, we study voltage distribution per node when the unit external current flows through the network of conducting links, which are systematically added in time and attached to the network with a preferential rule. We grow a network and let the constant current flow into last-added node and leave the graph at its root (first node), which is kept at zero potential. Compared to the static case, in the emergent graph structure the time *when* a node was added to the graph determines *how* it will be connected. By subsequent addition of nodes both number of links grows and a new structure emerges among preexisting nodes, making the conduction path between last-added node and root fluctuate. These features affect conduction on an evolving network that we address in this work. To elucidate all aspects of the evolution, we simulate suitable random walks on networks of several sizes N, *i.e.*, after $t = N = 2^k \times 500$ added nodes, $k = 0, \cdots, 4$, and determine the universal voltage curve that solely depends on time when the node was added to the graph. For the graphs with tree structure we develop a mean-field theory which qualitatively describes the numerical data.

Starting from the first node, the network grows by addition of one node and one link per time step. The link is directed to one of preexisting nodes s with the probability $p_{in}(s,t) = \frac{\alpha + q_{in}(s,t)}{(1+\alpha)t}$, where $\alpha \leq 1$ is a parameter and $q_{in}(s,t)$ is the number of in-links of the recipient node s at the moment t. Similarly, in the general case an out-link at time t occurs from new added node with probability g, whereas with $1-g$ it is a rewiring link from an earlier node, which is selected with probability [5] $p_{out}(s,t) = \frac{\alpha + q_{out}(s,t)}{(1+\alpha)t}$. By solving the corresponding rate equations with the right boundary conditions $q_{in}(s, s=t) = 0$, and

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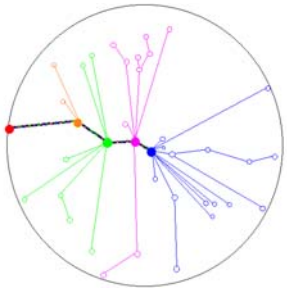

Fig. 1. A network growing with the preference rule shown after 39 steps. Distance of a node (small circle) from the very first node in the center illustrates the moment of its addition to the network. Current flows along the conduction path (bold line) from the most recent node on the left towards the center. The clusters of nodes meet the conduction path at junction nodes which are shown by bullets.

$q_{out}(s, s = t) = g$, we have that the number of links per node increases in time as

$$q_\kappa(s,t) = A_\kappa \left[\left(\frac{t}{s} \right)^{\gamma_\kappa} - B_\kappa \right] . \qquad (1)$$

Here index κ means 'in' or 'out' and the corresponding constants are $A_{in} = \alpha$, $B_{in} = 1$, $A_{out} = \alpha + g$, $B_{out} = \alpha/((\alpha + g)$, and the exponents $\gamma_{in} = 1/(1 + \alpha)$ and $\gamma_{out} = (1 - g)/(1 + \alpha)$, respectively. Note that for $g < 1$ rewiring among the preexisting nodes occurs, which is the mechanism that leads to the hierarchical structure of out-links, as demonstrated for instance in the model of the world-wide Web [5]. In this case a number of closed cycles on the graph occurs. For $g = 1$, however, we are left with tree structure of the graph and equation (1) for out-links reduces to $q_{out}(s,t) = \alpha$ at all nodes in the network. In what follows we will mainly discuss the case $g = 1$ and $\alpha = 1$ where we have [7] $q(s,t) = q_{in}(s,t) + 1 = \left(\frac{t}{s}\right)^{1/2}$. We grow an ensemble of networks with these parameters on computer (an example with first 39 nodes is shown in Fig. 1). The computed average number of links $q(s,t)$ per node after $t = N = 8 \times 10^3$ evolution steps is shown as top curve in Figure 2, which agrees well with the exact expression in equation (1). As the network grows we fix the elements of the adjacency matrix $\hat{A}$, so that after N steps we have an $N \times N$ matrix with elements $a_{xy} = 1$ when a link $x \to y$ occurs, and zero otherwise. Here we assume that these links are conducting in both directions.

An electrical network can be regarded as a graph in which the resistance R_{xy} is associated to the edge (link) between each pair of connected nodes $x \to y$. When two points (nodes) a and b of the graph are connected to poles of a battery, the current and the voltage in the interior of the graph are governed by the Kirchhoff's laws. In particular, when the potential difference occurs between points x and y, the current is given by the Ohm's law $i_{xy} = (V_x - V_y)C_{xy}$, where $C_{xy} = 1/R_{xy}$ is the conductance of the respective link. By the Kirchhoff's current law total current outflow from any point in the interior is

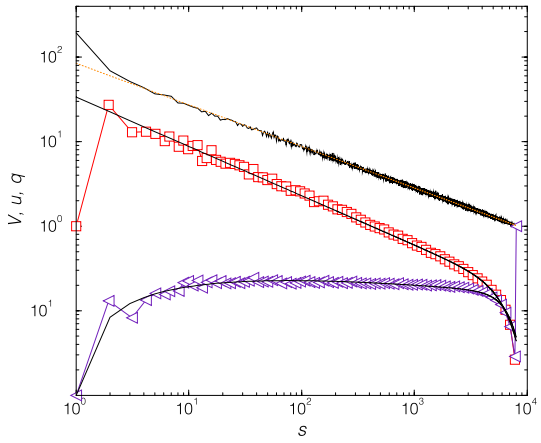

Fig. 2. Average total degree $q(s,N) = 1 + q_{in}(s,N)$ per node (top), number of visits $u(s,N)$ (middle) and voltage $V(s,N)$ (bottom) per node against time s when the node was added to the network with $N = 8000$ added nodes. Also shown are exact result $q(s,N) = (N/s)^{1/2}$ (dotted line) and fits $u(s,N) = 33.6 s^{-0.58} \exp[-1.4(s/N)^3]$, and $V(s,N) = 0.142 \ln(s) s^{-0.23} \exp[-1.3(s/N)^4]$. Exact value $V(s = 1) = 0$ was moved to a finite 10^{-2} to enable presentation on the logarithmic scale. Data for $u(s,N)$ and $V(s,N)$ normalized to total number of walkers and log-binned.

zero, $\sum_y i_{xy} = 0$, we then find for the voltage

$$V_x = \sum_y V_y C_{xy}/C_x. \qquad (2)$$

where $C_x = \sum_y C_{xy}$ and the sum is over all nodes y which are connected to x.

The averaging property expressed by equation (2) implies that the voltage is a *harmonic* function on the interior points of the graph. This makes the basis for the probabilistic interpretation of the voltage [4,8]. Namely, one can define another harmonic function, *e.g.*, by using the random walk [9] on the graph, with the same boundary values. The random walk determined by the electrical network is defined as an (ergodic reversible) Markov chain with the transition probabilities P_{xy} that are weighted with the conductances as $P_{xy} = C_{xy}/C_x$. Then, when the constant voltage is applied to the graph such that $V_a = 1$ and $V_b = 0$, the voltage in an interior point x is determined as the hitting probability h_x that a walker staring at x reaches the point a before reaching b. In the scenario, which we also use in this work, when a constant current flows into the network at the point a and leaves at b the walk begins at a and is trapped when it reaches point b. The harmonic function which is equivalent to the voltage is then given by [4,8] $V_x = u_x/C_x$, where $u_x = \sum_y u_y P_{yx}$ is the expected number of visits of the walker to point x before it reaches b. Consequently, the current between interior points x and y is given by the *net* number of walks along the link between these two points.

In the network evolving for $t = N$ steps we apply the unit current flowing into the network at the last-added node ($a \equiv t$) and leaving it at the first-added node ($b \equiv 1$). We assume that all resistances are equal $R_{xy} = 1$ and the

walker moves both along out-links and against in-links with equal probability. Therefore $C_x = q(x,t)$, the total number of links attached to node x at time t. Hence the voltage at node x is

$$V(x,t) = \langle u(x,t)/q(x,t) \rangle, \tag{3}$$

where $u(x,t)$ is the number of visits at x made by the walkers starting at last node t *before* they are trapped at node 1. The averaging in equation (3) is over entire ensemble of walkers. As mentioned above, we simulate random walks after $t = N = 2^k \times 500$, $k = 0, 1, \cdots, 4$ evolution steps. We generate 400 different networks of the same size N and use 200 walkers at each network realization, hence the voltage in equation (3) is determined using 80 000 walkers. (Note that the simulated voltage appears to be normalized by a constant $L \sim \ln(N)$ related to the average length of the conduction path.) The results for the network after $N = 8000$ added nodes are given in Figure 2.

The power-law dependence of the average degree $q(s,N)$ on addition time s of a node (*cf.* Fig. 2 and Eq. (1)) manifests the basic property of the evolving networks with emergent scale-free structure, where the most connected nodes are those added to the network at earlier stages of the growth. It appears that the average number of visits per node also exhibits a power-law decay with the time of addition s with an exponent $\theta_u \approx 0.58(3)$ (see Fig. 2). A sharp exponential cut-off at recently added nodes $s \leq N$ suggests lack of links, that will appear only in later stages as the network continues to grow. We find the same finite-size effect on the voltage curve. However, the power-law dependence here is modified by a logarithmic term. Apart from the last point $s = N$ where (normalized) voltage is $V(N,N) = 1$, the approximate expression fits the data for $1 \leq s \leq N-1$ as (see Fig. 3)

$$V(s, t = N) = D(N) \ln(s) s^{-\theta} \exp\left[-1.3(s/N)^4\right]. \tag{4}$$

Here $D(N) \sim 1/\ln(N)$ and $\theta \approx 0.25 \pm 0.04$ for the range of network sizes N used in this simulations. In Figure 3 are shown separate fits for several simulated network sizes N. The finite-size (finite evolution time) effects can be adequately dealt with by rescaling of the respective curves by $f(s/N) = \exp\left[1.3(s/N)^4\right]/\ln(N)$. The master curve representing the scaled voltage as function of s is shown in Figure 3 (top panel).

The curvature of the universal voltage curve at early nodes and a subsequent decay with the addition time s can be related to the growth process as follows. Due to the preference linking often a direct link from the high-voltage node N is attracted by the group of nodes near the root, thus 'pumping' voltage to these nodes. (Note that in the absence of loops voltage decays linearly with the number of junctions along the conduction path, while the actual position of the conduction path fluctuates with node addition). On the other hand, the increase of the voltage is compromised by highly probable linking of the early nodes to the root node 1. For large evolution times the cluster of nodes having a path to node 1 grows faster compared to clusters linked to other junction points along

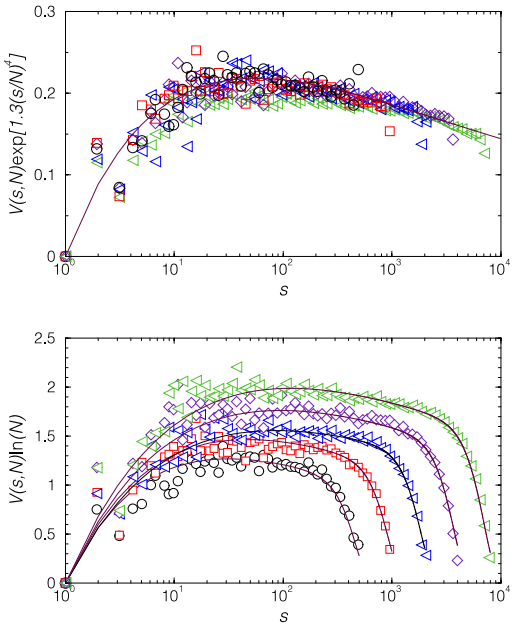

Fig. 3. Lower panel: Average voltage per node $V(s,N) \times \ln(N)$ *vs.* addition time $s < N$ obtained by random walks on networks after $N = 500$, 1000, 2000, 4000, and 8000 added nodes (bottom to top). Full lines: respective fits according to equation (4). Top panel: Scaled voltage per node, normalized by $\overline{D(N)}$. Full line: $y = 0.02(1-1/s) + 0.135 \ln(s) s^{-0.25}$. Data are log-binned.

the conduction path (see Fig. 1). Hence for large $s < N$ the probability increases for a node s to belong to the dominant cluster, which has zero voltage.

In the mean-field approach a network of t nodes can be regarded as consisting of $M \approx \ln(t)$ layers of nodes, where each layer is defined by the distance from the first node in the origin $d(x \to 1) = i = 1, 2, \cdots M$. Here the distance $d(x \to 1)$ is defined by number of links separating a node x and node 1. The conduction path from last added node to the origin cats through these layer making one junction point j_i at each layer. It is clear that a node which belongs to layer i is linked to a node on preceding layer $i-1$. However, while the network grows a node added at time t may be attached to one of the already existing layers. Hence, the population of layers R_i grows in time following precisely the above biased attachment rule, which leads to the rate equation [10]

$$\frac{\mathrm{d}R_{i+1}}{\mathrm{d}t} = \frac{(R_{i+1}/R_i + \alpha)R_i}{t(1+\alpha)}, \tag{5}$$

with the initial condition $R_0 = 1$. The system (5) can be solved recursively yielding

$$R_i(t) = (-\alpha)^i + t^\chi \sum_{\ell=0}^{i-1} \frac{K_{i-\ell}}{\ell!} \left(\frac{\alpha \ln(t)}{1+\alpha}\right)^\ell, \tag{6}$$

where $\chi = 1/(1+\alpha) = 1/2$ in the present case. The condition that $\sum_{i=0}^{m} R_i(t) = t = \mathrm{e}^m$ at current moment of time t, leads to the recursion relation between the coefficients. For instance, for $\alpha = 1$ we have $K_m = \mathrm{e}^{m/2} - \mathrm{e}^{-m/2}(1 + (-1)^m)/2 - \sum_{\ell=1}^{m-1} K_{m-\ell} \sum_{\kappa=0}^{\ell} (m/2)^\kappa/\kappa!$.

Now consider a subgraph G_s at the moment $s < t$ of the graph G_t grown for t steps. The number of nodes in the subgraph G_s, $x \leq s$ that are on the distance strictly larger than a given distance i, $d(x \to 1) > i$, is $n(d(x \to 1) > i) = s - \sum_{k=0}^{i} R_k(s)$. Then the probability that the node added at the moment s is among them reads $Prob(d(s \to 1) > i) = 1 - \sum_{k=1}^{i} R_k(s)/s$. In addition, if that node is on a path that hits the conduction path at a junction *above* layer i, then it has the voltage $V_s > i$. Then the probability of voltage $V_s > i$ is

$$Prob(V_s > i) = \left(1 - \sum_{k=1}^{i} R_k(s)/s\right) / R_{i+1}(t) \ , \qquad (7)$$

where, for short, $V_s \equiv V(s,t)$. Notice that the conduction path is set by addition of the last node t in the entire graph. Therefore the probability $1/R_{i+1}(t)$ that the path from s to 1 does not miss the junction point on $(i+1)$th layer depends on the population of that layer at the moment t. Combining the two probabilities we have

$$Prob(V_s = i) = Prob(V_s > i-1) - Prob(V_s > i) \ , \qquad (8)$$

and the average voltage is given by $V_s = \sum_{i=0}^{M} i Prob(V_s = i)$. Expanding the sum and using the corresponding expressions for $R_i(s)$ for $\alpha = 1$ we find $V_s \approx c_0(1 - 1/s) + c_1 s^{-1/2} \ln(s) + c_2 s^{-1/2} (\ln(s))^2 + \cdots$. Here c_k depend on number of nodes t. In particular, c_0 decreases for large t and additional higher-order terms $\ln(s)^k$ appear. This series contributes to the effectively reduced exponent of s below 1/2, that justifies the approximate fitting expression given in caption to Figure 3 (top). More details will be given elsewhere [10].

The simulated probability distribution of voltage obtained by the random-walk statistics is shown in Figure 4. It can be fitted with a stretched-exponential function. The probability distribution of survival time t_w of the random walk before trap, $P(t_w)$, and of the frequency of visit u of walkers to a given node, $P(u)$, appear to have power-law dependences with cut-offs, the latter resembling closely the topology of shortest paths on the graph [11].

We have demonstrated how the electrical properties of an emergent graph are shaped by the nature of linking processes governing growth of the network. This establishes technologically relevant link between, *e.g.*, conduction of complex digital circuits and the way that they are grown. The main results are summarized in the dependence of the voltage distribution per node inside the graph on the addition time of the node to the network. Our mean-field theory qualitatively describes the numerical data for the scale-free graphs with tree structure considered in this work. On the level of the random walk with a trap, which is determined by the electrical network, the observed voltage distribution can be related to power-law dependences of the wandering time of the walk and number of visits to a given node. As a side result we have shown that the universal scaling exponent of the distribution of visits of our "charged" random walk, which uses local navigation rules, coincide with the ones of the distribution of shortest paths through a node, for which costly global navigation is necessary. Thus, close relationship between minimal path and conduction path on an electrical network suggests potential use of "charged" random walk to determine global topological properties by using local search algorithms only. The numerical method is applicable to graphs of any link complexity, for instance as given by equation (1).

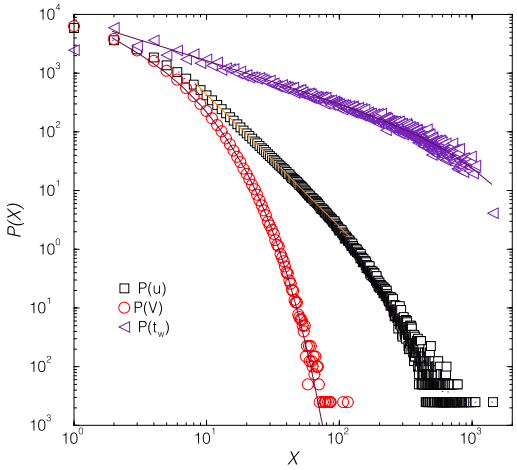

Fig. 4. Probability distribution $P(X)$ of voltage $X \equiv V$, and of number of visits u and elapsed time before trap t_w of random walkers on the network with $N = 10^3$ nodes. Solid lines: fits $P(V) = P_0 V^{-0.8} \exp(-V^{0.9}/3.8)$, and $P(t_w) = P_0 t_w^{-0.75} \exp(-t_w/N)$, with $P_0 = 11 \times 10^3$ and a power-law fit of the part of P(u) curve with slope 2.25±0.015.

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